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The estimation of space and time dependent strength of a volumetric heat source in a one-dimensional plate

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Abstract—An inverse analysis utilizing the conjugate gradient method of minimization and the adjoint equation is used to estimate the space and time dependent strength of a volumetric heat source with no prior information for the functional forms of timewise and spatial variations of the source strength. Simulated experimental data, needed for the inverse analysis, are generated by adding random errors to the calculated exact temperatures for the boundaries and interior of the plate. In order to examine the accuracy of estimations under the most strict conditions, test cases such as sawtooth shaped spatial variation of the source strengths are considered. The effects of the number and location of temperature sensors on the accuracy of the estimations are systematically examined. The estimates are significantly improved when the thermocouples are positioned close to the locations where the source strength exhibits peaks and valleys.

1. INTRODUCTION

In the field of heat transfer, the inverse analysis has been widely used for the examination of surface conditions such as temperature or heat flux distributions, and thermal properties such as thermal conductivity and heat capacity of solids, by utilizing the transient temperature measurements taken within the medium [1–7]. However, the inverse problem of simultaneously estimating the spatial and timewise variation of the strength of a volumetric heat source appears to have received little attention.

Inverse heat conduction problems are known to be ill-posed [1–3], in contrast to the direct heat conduction problems, which are well-posed, that is, the solution exists, the solution is unique and the solution is stable to small changes in the input data. A variety of numerical and analytical techniques have been proposed for the solution of such problems. They include, among others, the least squares method modified by the addition of a regularization term [1, 2], the sequential estimation approach [1, 4–6] and more recently the conjugate gradient method [7–11] using an adjoint equation. In the area of identifying the dynamic behavior of linear systems, the inverse problems have been recast as an ill-posed Volterra integral equation of convolution type [12–14].

In this work we estimate the space and time dependent strength of a volumetric heat source, which releases its energy continuously inside a one-dimensional plate, by an inverse analysis using the conjugate gradient method with an adjoint equation. No prior information is used on the functional forms of timewise and spatial variations of the strength of the heat source.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

A one-dimensional plate of thickness L, initially at a uniform temperature T_i contains a volumetric energy source of unknown strength $g(x, t)$, W m^{-3}. For times $t > 0$, energy is generated by the source at an unknown rate and spatial distribution, while the boundaries of the plate are kept insulated.

Our objective is to estimate the unknown strength of the heat source, $g(x, t)$, from the transient temperature recordings taken at the boundaries and interior of the plate.

The mathematical formulation of this transient heat conduction problem for the case when the source strength $g(x, t)$ is known will be referred to as the *direct problem*. It is given in the dimensionless form as

$$\frac{\partial^2\Theta(X,\tau)}{\partial X^2}+G(X,\tau)=\frac{\partial\Theta(X,\tau)}{\partial\tau},\quad \text{in } 0<X<1,\quad \tau>0 \tag{1a}$$

$$\frac{\partial\Theta}{\partial X}=0 \quad \text{at } X=0,\quad \text{for } \tau>0 \tag{1b}$$

$$\frac{\partial\Theta}{\partial X}=0 \quad \text{at } X=1,\quad \text{for } \tau>0 \tag{1c}$$

$$\Theta(X,\tau)=0 \quad \text{for } \tau=0, \text{ in the region } 0\leqslant X\leqslant 1 \tag{1d}$$

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NOMENCLATURE

G	dimensionless strength of the heat source
J	functional defined by equation (3)
J'_G	gradient of the functional J
M	total number of sensors
P^k	direction of descent, equation (12)
X	dimensionless spatial variable
X_m	dimensionless sensor location ($m = 1, 2, \ldots, M$)
Z_m	dimensionless simulated measured temperatures.

Greek symbols

β^k	step size, equation (14)
γ^k	conjugate coefficient, equation (15)
δ	Dirac delta function
ε	tolerance for the stopping criterion, equation (17)
Θ	computed dimensionless temperature
Θ_m	computed dimensionless temperature at the sensor location
$\lambda(X, \tau)$	adjoint function defined by equations (6)
σ	standard deviation of temperature measurement errors
τ	dimensionless time
τ_f	dimensionless final time.

where the dimensionless quantities include Θ the temperature, X the spatial variable, $G(X, \tau)$ the source strength and τ the time.

The physical significance of the inverse problem considered here is as follows. Suppose the source strength $G(X, \tau)$ is not known, instead temperature measurements, taken at both boundaries and interior of the plate, are available as a function of time. By utilizing these measured temperatures, determine the space and time dependent strength of the heat source $G(X, \tau)$.

3. THE INVERSE ANALYSIS

The solution of the inverse problem described previously with the conjugate gradient method involves the following basic steps: (a) the direct problem; (b) the sensitivity problem; (c) the adjoint problem and the gradient equation; (d) the conjugate gradient method of minimization; and (e) the stopping criterion. After describing the computational procedure in each of these steps, a solution algorithm will be presented for the determination of the unknown source strength $G(X, \tau)$. It is to be noted that the order of presentation of the various steps listed above will not necessarily follow the same order as that in the solution algorithm. We present below the mathematical description of these basic steps.

3.1. *The direct problem*

The direct problem, as stated previously, refers to the solution of system (1) when the source strength $G(X, \tau)$ is known. In the inverse problem this quantity is unknown and determined by the solution algorithm that will be described in Section 3.6.

3.2. *The sensitivity problem*

The sensitivity problem is obtained by replacing in the direct problem (1), $\Theta(X, \tau)$ by $\Theta(X, \tau)+\Delta\Theta(X, \tau)$, $G(X, \tau)$ by $G(X, \tau)+\Delta G(X, \tau)$ and subtracting from the resulting expression the original direct problem (1), where $\Delta\Theta(X, \tau)$ and $\Delta G(X, \tau)$ are small perturbations. We find

$$\frac{\partial^2[\Delta\Theta(X,\tau)]}{\partial X^2} + \Delta G(X,\tau) = \frac{\partial[\Delta\Theta(X,\tau)]}{\partial \tau} \quad \text{in } 0 < X < 1, \quad \tau > 0 \qquad (2a)$$

$$\frac{\partial(\Delta\Theta)}{\partial X} = 0 \quad \text{at } X = 0, \quad \text{for } \tau > 0 \qquad (2b)$$

$$\frac{\partial(\Delta\Theta)}{\partial X} = 0 \quad \text{at } X = 1, \quad \text{for } \tau > 0 \qquad (2c)$$

$$\Delta\Theta(X,\tau) = 0 \quad \text{for } \tau = 0, \text{ in the region } 0 \leqslant X \leqslant 1. \qquad (2d)$$

The above equations contain a new unknown quantity $\Delta G(X, \tau)$ which is determined as described in step 7 of the solution algorithm in Section 3.6.

3.3. *The adjoint problem and the gradient equation*

The inverse problem is solved as an optimization problem by requiring that the unknown function $G(X, \tau)$ minimize the functional $J[G(X, \tau)]$ defined by

$$J[G(X,\tau)] \equiv J = \int_0^{\tau_f} \sum_{m=1}^{M} [\Theta_m(\tau) - Z_m(\tau)]^2 \, d\tau \qquad (3)$$

where τ_f is the final time, M is the number of sensors, and $\Theta_m(\tau)$ and $Z_m(\tau)$ are the dimensionless computed and measured temperatures, respectively, at each sensor location as a function of time.

The adjoint problem is developed by multiplying equation (1a) by the *adjoint function* $\lambda(X, \tau)$, integrating the resulting expression over time and space domain and then adding the result to the functional given by equation (3). We obtain

$$J = \int_0^{\tau_f} \sum_{m=1}^{M} [\Theta_m(\tau) - Z_m(\tau)]^2 \, d\tau + \int_0^{\tau_f} \int_0^1 \lambda(X,\tau) \left[\frac{\partial^2\Theta}{\partial X^2} + G(X,\tau) - \frac{\partial\Theta}{\partial\tau} \right] dX \, d\tau. \qquad (4)$$

Note that when Θ is the exact solution of problem (1), the second term on the right-hand side of this equation vanishes and we recover equation (3).

The variation ΔJ of the functional J is obtained by perturbing $G(X, \tau)$ by $\Delta G(X, \tau)$ and $\Theta(X, \tau)$ by $\Delta\Theta(X, \tau)$ in equation (4), and subtracting from it the original equation (4). Neglecting the second order terms, we obtain

$$\Delta J = \int_0^{\tau_f}\int_0^1 \sum_{m=1}^{M} 2[\Theta_m(\tau)-Z_m(\tau)] \times \Delta\Theta\delta(X-X_m)\,\mathrm{d}X\,\mathrm{d}\tau + \int_0^{\tau_f}\int_0^1 \lambda(X,\tau) \times \left[\frac{\partial^2(\Delta\Theta)}{\partial X^2}+\Delta G(X,\tau)-\frac{\partial(\Delta\Theta)}{\partial \tau}\right]\mathrm{d}X\,\mathrm{d}\tau \quad (5)$$

where δ is the Dirac delta function and X_m are the sensor locations.

The right-hand side of this expression is integrated by parts, the boundary and initial conditions of the sensitivity problem are utilized, and in the resulting equation, the coefficients of $\Delta\Theta$ are required to vanish. The following adjoint problem is obtained for the determination of the *adjoint function* $\lambda(X, \tau)$:

$$\frac{\partial^2\lambda(X,\tau)}{\partial X^2}+\sum_{m=1}^{M} 2[\Theta_m(\tau)-Z_m(\tau)]\delta(X-X_m) = -\frac{\partial\lambda(X,\tau)}{\partial\tau} \quad (6a)$$

$$\frac{\partial\lambda}{\partial X}=0 \quad \text{at } X=0 \quad (6b)$$

$$\frac{\partial\lambda}{\partial X}=0 \quad \text{at } X=1 \quad (6c)$$

$$\lambda = 0 \quad \text{for } \tau=\tau_f \quad (6d)$$

and the following integral term is left

$$\Delta J[\Delta G(X,\tau)] = \int_0^{\tau_f}\int_0^1 \lambda(X,\tau)\Delta G(X,\tau)\,\mathrm{d}X\,\mathrm{d}\tau. \quad (7)$$

By the definition of gradient the following relation holds [9, 15, 16]:

$$\Delta J = \int_0^{\tau_f}\int_0^1 J'_G(X,\tau)\Delta G(X,\tau)\,\mathrm{d}X\,\mathrm{d}\tau. \quad (8)$$

A comparison of equations (7) and (8) reveals that the gradient of the functional, $J'_G(X, \tau)$, is given by

$$J'_G(X,\tau) = \lambda(X,\tau). \quad (9)$$

The adjoint problem (6) is different from the standard initial value problems in that the final time condition at time $\tau = \tau_f$ is specified instead of the customary initial condition $\tau = 0$. However, problem (6) can be transformed to an initial value problem by introducing a new time variable τ^* defined as

$$\tau^* = \tau_f - \tau. \quad (10)$$

Then the standard solution techniques can be applied for the solution of the transformed initial value problem.

3.4. *The conjugate gradient method of minimization*

We consider the following iterative procedure for the estimation of the unknown strength of the heat source, $G(X, \tau)$, given in the form [8, 17, 18]

$$G(X,\tau)^{k+1} = G(X,\tau)^k - \beta^k P(X,\tau)^k, \quad k=0,1,2,\ldots \quad (11)$$

where β^k is the *step size* in going from step k to step $k+1$ and $P(X, \tau)^k$ is the *direction of descent* at step k defined as [17]

$$P(X,\tau)^k = J'_G(X,\tau)^k + \gamma^k P(X,\tau)^{k-1} \quad \text{with } \gamma^0 = 0 \quad (12)$$

where γ^k is the conjugate coefficient. We note that the special case $\gamma^k = 0$ corresponds to the steepest descent.

The step size β^k appearing in equation (11) is determined by minimizing the functional $J[G(X, \tau)^{k+1}]$ given by equation (3), with respect to β^k, i.e.

$$\min_{\beta^k} J[G(X,\tau)^{k+1}] = \min_{\beta^k}\int_0^{\tau_f}\sum_{m=1}^{M}[\Theta_m(G^k-\beta^k P^k)-Z_m(\tau)]^2\,\mathrm{d}\tau. \quad (13)$$

Applying a Taylor series expansion and then minimizing with respect to β^k the following result is obtained:

$$\beta^k = \frac{\sum_{m=1}^{M}\int_0^{\tau_f}[\Theta_m(G^k)-Z_m]\Delta\Theta(P^k)\,\mathrm{d}\tau}{\sum_{m=1}^{M}\int_0^{\tau_f}[\Delta\Theta(P^k)]^2\,\mathrm{d}\tau}. \quad (14)$$

Different definitions of the conjugate coefficient γ^k are reported in the literature [7, 8]. In the present work we used the following expression [17, 18]

$$\gamma^k = \frac{\int_0^{\tau_f}\int_0^1 [J'(G^k)]^2\,\mathrm{d}X\,\mathrm{d}\tau}{\int_0^{\tau_f}\int_0^1 [J'(G^{k-1})]^2\,\mathrm{d}X\,\mathrm{d}\tau}, \quad k=1,2,\ldots \quad (15)$$

3.5. *The stopping criterion*

As measurement errors are always present in practical applications, the discrepancy principle [7, 17], described below, is used to establish the criterion for stopping the iterations in the estimation of the strength of the heat source.

Let the standard deviation σ of the measurement errors be the same for all sensors and measurements, that is

$$\Theta_m(\tau)-Z_m(\tau) \cong \sigma. \quad (16)$$

Introducing this result into equation (3) we obtain

$$\int_0^{\tau_f} \sum_{m=1}^{M} \sigma^2 \, d\tau = M\sigma^2\tau_f \equiv \varepsilon^2. \tag{17}$$

The value of ε thus established is used as the stopping criterion, that is

$$J[G(X,\tau)^{k+1}] < \varepsilon^2. \tag{18}$$

3.6. *The solution algorithm*

The computational procedure described above is summarized in the following algorithm:

Step 1. Choose an initial guess $G(X,\tau)^0$, for example $G(X,\tau)^0$ = constant.

Step 2. Solve the direct problem given by equations (1), to obtain $\Theta(X,\tau)$.

Step 3. Knowing the computed temperature $\Theta_m(\tau)$ and the measured temperature $Z_m(\tau)$, at the sensor locations, solve the adjoint problem (6), and obtain $\lambda(X,\tau)$.

Step 4. Knowing $\lambda(X,\tau)$ compute the gradient function $J'_G(X,\tau)$ from equation (9).

Step 5. Compute the conjugate coefficient γ^k from equation (15).

Step 6. Compute the direction of descent $P(X,\tau)^k$ from equation (12).

Step 7. Setting [9] $[\Delta G(X,\tau)]^k = P(X,\tau)^k$ solve the sensitivity problem (2) to obtain $\Delta\Theta(X,\tau)$.

Step 8. Compute the step size in going from step k to step $k+1$, β^k, from equation (14).

Step 9. Compute $G(X,\tau)^{k+1}$ from equation (11).

Step 10. Terminate the iterations when the stopping criterion, given by equation (18) is satisfied. If not, go to Step 2.

4. RESULTS AND DISCUSSION

The accuracy of the inverse analysis for estimating the space and time dependent strength of a volumetric heat source is examined for test cases by using simulated transient temperature readings, $Z_m(\tau)$.

The functions exhibiting abrupt changes are generally the most difficult cases to recover with inverse analysis. In order to perform the tests under such strict conditions, we considered, among others, a sawtooth shaped function for the spatial distribution of the source strength.

The simulated transient temperature data, $Z_{m,i}$, containing measurement errors are generated by adding random errors to the computed exact temperatures, $\Theta_{m,i}$, as

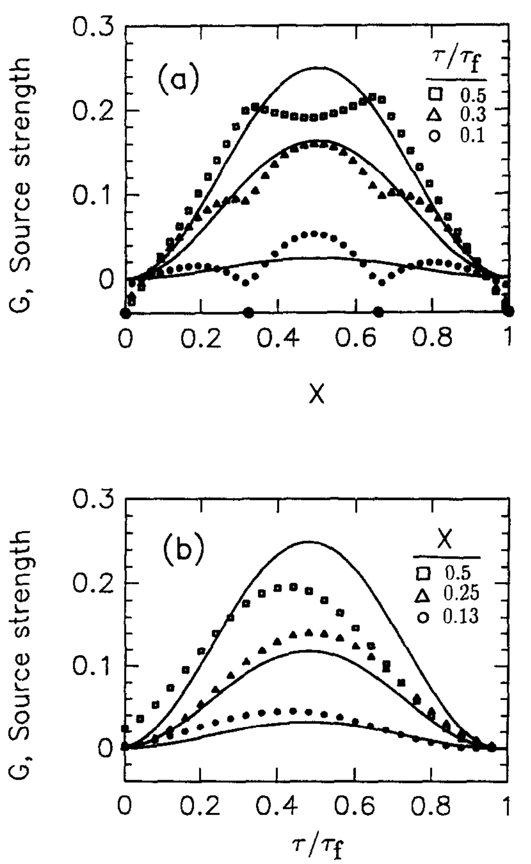

FIG. 1. The estimation of a space and time dependent volumetric heat source using four temperature sensors and $\sigma = 0$. (a) Spatial variation. (b) Timewise variation.

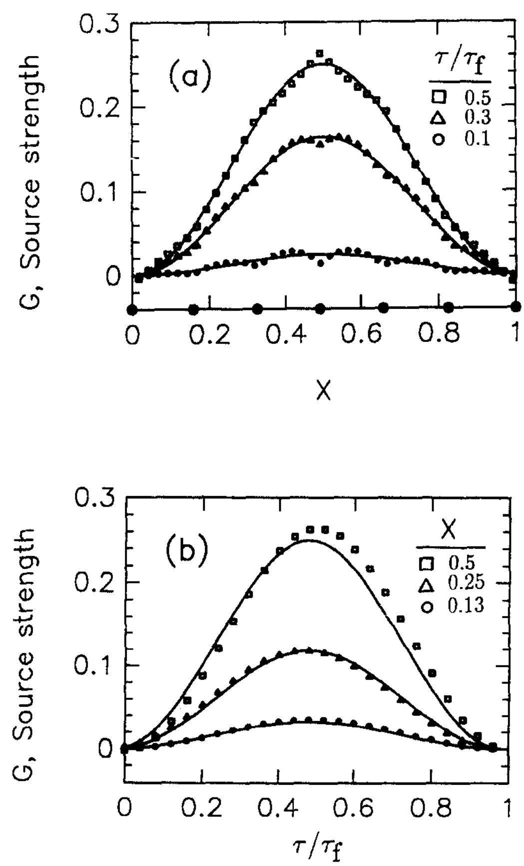

FIG. 2. The estimation of a space and time dependent volumetric heat source using seven temperature sensors and $\sigma = 0$. (a) Spatial variation. (b) Timewise variation.

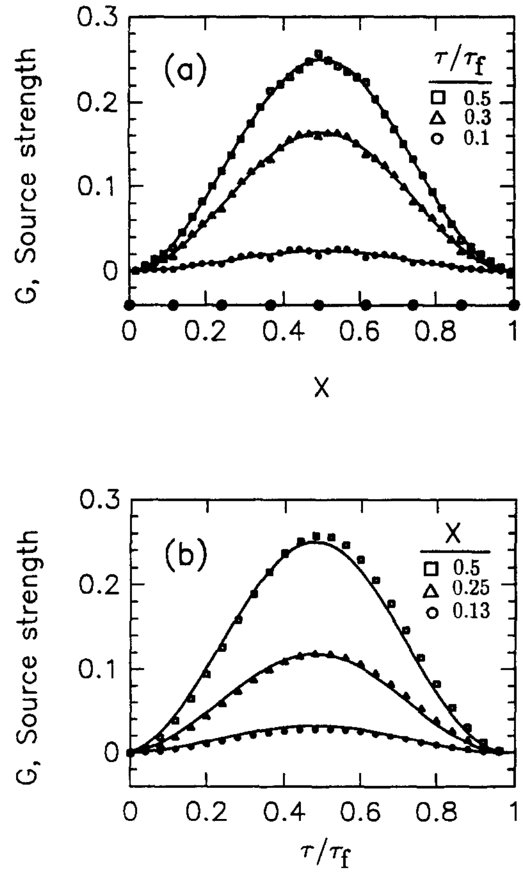

FIG. 3. The estimation of a space and time dependent volumetric heat source using nine temperature sensors and $\sigma = 0$. (a) Spatial variation. (b) Timewise variation.

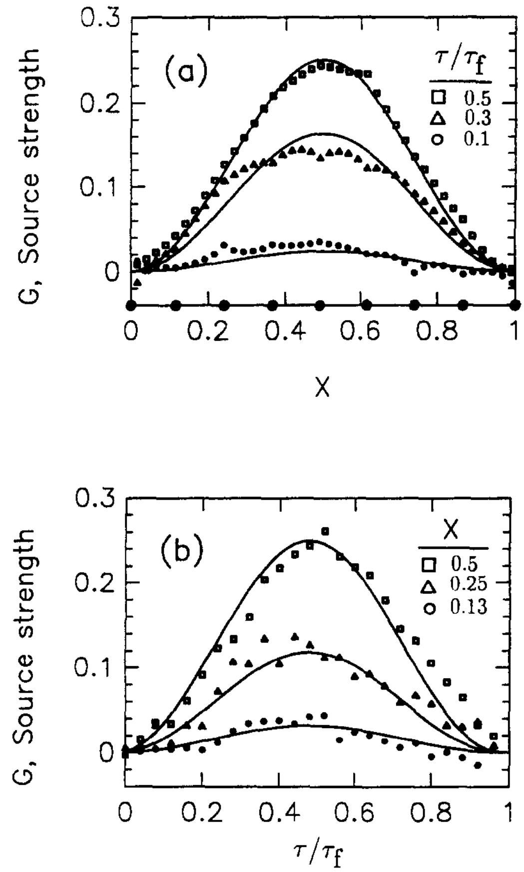

FIG. 4. The estimation of a space and time dependent volumetric heat source using nine temperature sensors and $\sigma = 0.05$. (a) Spatial variation. (b) Timewise variation.

$$Z_{m,i} = \Theta_{m,i} + \sigma e_{m,i} \quad i = 1, 2, \ldots, N, \quad m = 1, 2, \ldots, M \tag{19}$$

where M refers to the total number of sensors and N to the total number of measurements taken with each sensor. In addition, σ is the standard deviation of measurement errors which is assumed to be the same for all measurements and $e_{m,i}$ is the normally distributed random error generated by the IMSL subroutine RNNOR [19]. For normally distributed random errors, there is a 99% probability of the value of $e_{m,i}$ lying in the range

$$-2.576 < e_{m,i} < 2.576. \tag{20}$$

In this paper we examine the effects of the shape of the spatial variation of the heat source, the total number of temperature sensors and their location, on the accuracy of the estimations.

Figures 1–3 show the effects of the number of sensors on the estimation of the source strength, $G(X, \tau)$, for the case when the spatial and timewise variations are considered sinusoidal functions. The standard deviation of the measurement errors is taken zero (i.e. $\sigma = 0$).

Figure 1 shows the estimations with four thermocouples located at $X_m = 0.0$, 0.33, 0.67 and 1.0, while in Fig. 2 seven thermocouples are used at the locations $X_m = 0.0, 0.17, 0.33, 0.50, 0.67, 0.83$ and 1.0. Figure 3 is for the case involving nine thermocouples at locations $X_m = 0.0, 0.13, 0.25, 0.38, 0.5, 0.63, 0.75, 0.88$ and 1.0. The locations of the thermocouples are shown by dots along the X axis.

Figure 1 shows that the use of four thermocouples does not seem to be sufficient for accurate estimation of the spatial and timewise variation of the source strength. The use of seven thermocouples improves the estimations as shown in Fig. 2, while the use of nine thermocouples further improves the estimations as shown in Fig. 3.

We also tried a curve fit by cubic splines on the simulated measured temperatures, but it did not improve the estimations.

Figure 4 shows similar results for the case of standard deviation $\sigma = 0.05$. This case is to be compared to that shown in Fig. 3 for $\sigma = 0$. The standard deviation $\sigma = 0.05$ corresponds to an error of 13% for a dimensionless maximum measured temperature $Z_{max} = 1$. Even with such a large measurement error, the estimation is in good agreement with the exact test case.

Figure 5 is intended to show the effects of sharp peaks on the spacewise variation of the source strength while a sinusoidal variation is chosen for the time variation. The nine thermocouples locations used for this case are the same as those used in Figs. 3 and 4. A standard deviation $\sigma = 0.02$ chosen for this case corresponds to an error of 5%.

Figure 5(a) shows the estimations for the spatial variation of the source at three different times, i.e. $\tau/\tau_f = 0.5$, 0.3 and 0.1. Figure 5(b) shows the estimations for the timewise variation of the source strength at three different locations, i.e. $X = 0.25$, 0.17 and 0.08. The estimations are in good agreement with the exact strengths chosen for the test cases.

Figures 6 and 7 are intended to illustrate the effects of the location of the temperature sensors on the accuracy of the estimation of the source strength. In both cases seven thermocouples are used. For the case shown in Fig. 6 the thermocouples are located at $X_m = 0.0$, 0.08, 0.30, 0.50, 0.70, 0.92 and 1.0, which are away from the peaks and valleys, while in Fig. 7 they are located close to the peaks and valleys, i.e. $X_m = 0.0$, 0.17, 0.33, 0.50, 0.67, 0.83 and 1.0. A standard deviation of $\sigma = 0$ is used in both cases.

Figures 6(a) and 7(a) show the estimations for the spatial variation of the source strength at three different times, $\tau/\tau_f = 0.5$, 0.3 and 0.1. Figures 6(b) and 7(b) show the estimations for the timewise variation of the source strength at three different locations, i.e. $X = 0.17$, 0.12 and 0.06.

We note that in Fig. 7 thermocouples are located at the peaks and valleys of the spatial distribution of the source strength while in Fig. 6 the thermocouples are away from the peaks and valleys. As expected the estimations are good with the former. Therefore, if some prior information is available on the location of peaks and valleys, the location of the thermocouples can be arranged accordingly.

A mesh with 121 nodes in space and 50 in time was used to solve the direct, sensitivity and adjoint problems with finite difference. A dimensionless spatial spacing of $\Delta X = 1/120$ and a time step $\Delta\tau = 0.002$ were used in the computations. To illustrate the physical significance of the dimensionless time $\tau_f = 0.1$, we consider a plate of thickness $L = 0.2$ m, and thermal diffusivity $\alpha = 10^{-6}$ m^2 s^{-1}. For such a case, $\tau_f = 0.1$ corresponds to $t_f = 4000$ in real time.

The numerical computations required only a few minutes in the IBM RISC 6000 computer.

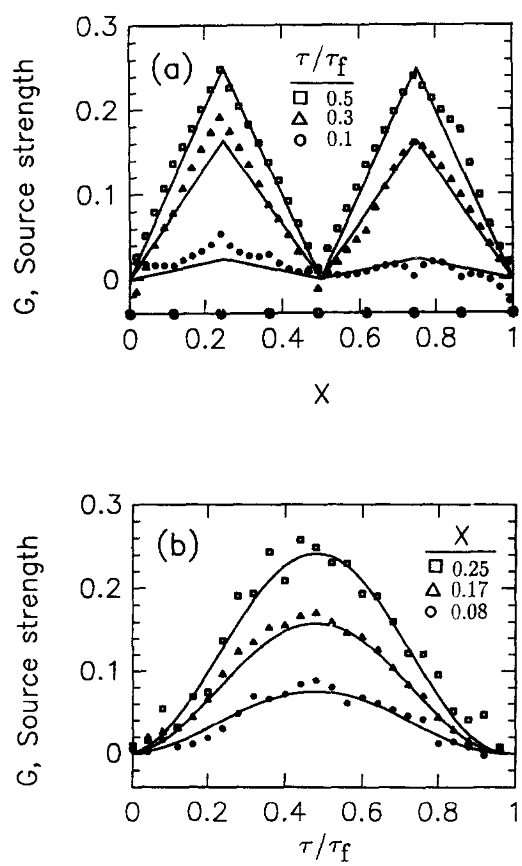

Fig. 5. The estimation of a space and time dependent volumetric heat source using nine temperature sensors and $\sigma = 0.02$. (a) Spatial variation. (b) Timewise variation.

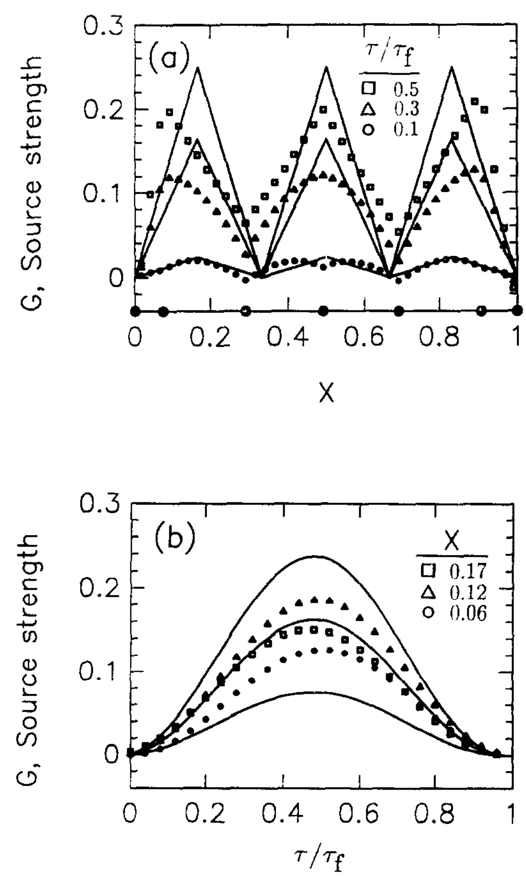

Fig. 6. The estimation of a space and time dependent volumetric heat source using seven temperature sensors poorly located and $\sigma = 0$. (a) Spatial variation. (b) Timewise variation.

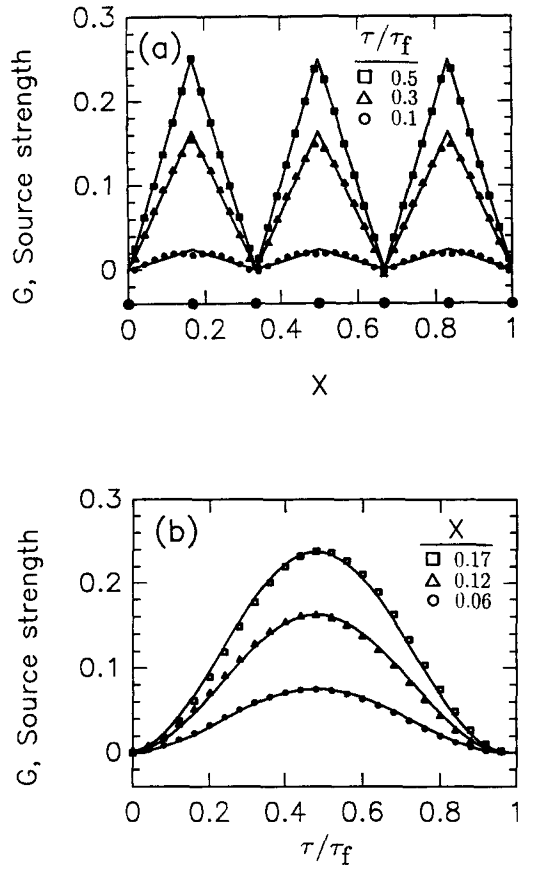

FIG. 7. The estimation of a space and time dependent volumetric heat source using seven temperature sensors well located and $\sigma = 0$. (a) Spatial variation. (b) Timewise variation.

5. CONCLUSIONS

The inverse problem of estimating the unknown space and time dependent strength of a volumetric heat source in a one-dimensional plate has been solved using the conjugate gradient method of minimization utilizing the adjoint equation approach. Several test cases involving different shapes, number of temperature sensors and their locations, were considered. When the spatial distribution of the source strength contains sharp peaks and valleys, the estimations can be improved if information is available on their location.

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